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COORDINATE PERTURBATION AND MULTIPLE SCALE IN GASDYNAMICS

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CONTENTS

Usually, the application of the coordinate perturbation technique consists in transforming the equations to perturbed coordinates, and determining from the transformed equations the amount of coordinate straining appropriate to obtain a uniformly valid expansion. However, the transformed equations may become unwieldy with increasing order of the system, number of variables, and order of the approximation. There exists a much simpler way of applying the technique, which bypasses the transformed equations and provides the appropriate coordinate stretching by simple algebraic manipulations on the non-uniformly valid expansion obtained by straightforward expansion from the original equations.

Interesting results are obtained by applying the procedure to two gasdynamical problems. In the first the flow field around a supersonic two-dimensional wing is determined up to third order, including a uniformly valid representation of the front shock shape, valid even when the shock does not start at the leading edge. The second problem concerns the oscillations in a closed tube following an arbitrary initial disturbance, both when the two ends are closed, and when one of the two ends contains an oscillating piston (the inviscid Chester problem). In both problems the uniformly valid expansions are substantially simpler than the non-uniformly valid. But most interesting is the result that the uniformly valid expansions cannot be obtained without supplementing the coordinate perturbation technique by the multiple scale technique.

1. A simple formulation

Among the techniques developed in recent years for the treatment of nonlinear p.d. equations the oldest is Lighthill's (1949, 1961) extension of the classical Poincaré method of nonlinear mechanics. This technique, which substantially consists in perturbing not only the unknowns, but also the independent variables, goes presently under several names, such as the 'PLK technique', the 'coordinate stretching – or straining – technique', and the 'coordinate perturbation technique'.

Although success has sporadically been achieved for other types of p.d. equations, this

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technique appears to be particularly well suited for equations of the hyperbolic type, where it makes it possible to follow to any order the actual wave propagation paths.

In spite of the fact that the idea on which the method is based is quite simple, its application following the standard procedure can become discouragingly cumbersome and confusing when many variables are involved, particularly for higher order approximations. Suppose indeed, to fix ideas, that we are seeking the solution $z(x, y; \epsilon)$ of a p.d. equation in two variables, containing a small parameter ϵ, and suppose that the result $z = z_0 + \epsilon z_1 + \epsilon^2 z_2 + \dots$, of a straightforward expansion is not uniformly valid for $\epsilon \to 0$. Then, substantially, the idea is to associate to the z development the additional developments

$$x = X + \epsilon x_1 + \epsilon^2 x_2 + \dots, \quad y = Y + \epsilon y_1 + \epsilon^2 y_2 + \dots$$

for the independent variables† and to consider z to be a function of the strained variables X, Y rather than of the original unstrained variables, determining the available straining coefficients $x_1, y_1, x_2, y_2, \dots$ (to be also considered functions of X, Y) in such a way as to restore the uniform validity of the z development. This is achieved, sometimes rather laboriously, by working on the p.d. equations transformed to the X, Y variables. For instance z_x (the subscript letter indicates partial differentiation) is replaced by

$$z_x = z_{0X} + \epsilon(z_{1X} + z_{0X} x_{1X} + z_{0Y} y_{1X}) + \epsilon^2[z_{2X} + z_{1X} x_{1X} + z_{1Y} y_{1X} + z_{0X}(x_{2X} + x_{1X}^2 + x_{1Y} y_{1X}) + z_{0Y}(y_{2X} + y_{1X} x_{1X} + y_{1Y} y_{1X})] + \dots$$

and z_{xx} in a similar way by an expression in terms of first- and second-order derivatives containing, to order ϵ^2, 27 terms. It is evident that the transformed equations become increasingly complicated and confusing (because the number of unknown functions increasingly exceeds the number of equations) with increasing order, number of variables and order of the approximation, with the result that the technique is seldom applied except to the simplest types of p.d. equations and to the lowest order of approximation.

In reality, to the simplicity of the idea corresponds a very simple way of applying the method, a way which – despite its being described in 1962 by Pritulo (1962) – has been all but ignored by the occidental experts.‡ It seems appropriate to reproduce here the substance, slightly generalized, of Pritulo's suggestion. For more generality, suppose that the equations contain, in addition to the small parameter ϵ, m unknown variables $z^{(i)}$ and n independent space and time variables (coordinates) $x^{(h)}$. In a straightforward expansion procedure the m-dimensional vector $\boldsymbol{z}(\boldsymbol{x}; \epsilon)$ with components $z^{(i)}$, function of ϵ and of the n-dimensional vector $\boldsymbol{x}$ with components $x^{(h)}$, is developed as

$$\boldsymbol{z}(\boldsymbol{x}; \epsilon) = \boldsymbol{z}_0(\boldsymbol{x}) + \epsilon \boldsymbol{z}_1(\boldsymbol{x}) + \epsilon^2 \boldsymbol{z}_2(\boldsymbol{x}) + \dots \tag{1.1}$$

and $\boldsymbol{z}_0, \boldsymbol{z}_1, \boldsymbol{z}_2, \dots$ are successively determined from the equations of increasing order. We now introduce a coordinate transformation from $x^{(h)}$ to $X^{(h)}(\boldsymbol{x}; \epsilon)$, and define the reciprocal transformation in vectorial form through the development

$$\boldsymbol{x}(\boldsymbol{X}; \epsilon) = \boldsymbol{X} + \epsilon \boldsymbol{x}_1(\boldsymbol{X}) + \epsilon^2 \boldsymbol{x}_2(\boldsymbol{X}) + \dots \tag{1.2}$$

the components of $\boldsymbol{X}$ being $X^{(h)}$, and $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots$ being vectors having for components the straining

† Lighthill actually considered only one such development. Lin (1954) did, however, take into consideration two.

‡ Only recently the author has become aware of a 1967 paper by Martin who independently proposed a method similar to Pritulo's, but less direct and more involved.

coefficients $x_1^{(h)}$, $x_2^{(h)}$, ... These represent n sets of functions, undefined for the moment, to be determined, if possible, in such a way as to insure the uniform validity of the development

$$z(x;\epsilon) + Z(X;\epsilon) = Z_0(X) + \epsilon Z_1(X) + \epsilon^2 Z_2(X) + \ldots \tag{1.3}$$

when the development (1.1) fails to be uniformly valid. But expanding in Taylor series we get

$$z(x;\epsilon) = z(X;\epsilon) + (x^{(h)} - X^{(h)})\, z_{x^{(h)}}(X;\epsilon) + \tfrac{1}{2}(x^{(h)} - X^{(h)})\,(x^{(k)} - X^{(k)})\, z_{x^{(h)}x^{(k)}}(X;\epsilon) + \ldots,$$

where we have used the summation convention with respect to the superscripts.

Introducing here the developments (1.1) and (1.2) and comparing with the development (1.3) we get

$$\left.\begin{aligned} Z_0(X) &= z_0(X),\\ Z_1(X) &= z_1(X) + x_1^{(h)}(X)\, z_{0,x^{(h)}}(X),\\ Z_2(X) &= z_2(X) + x_1^{(h)}(X)\, z_{1,x^{(h)}}(X) + x_2^{(h)}(X)\, z_{0,x^{(h)}}(X)\\ &\quad + \tfrac{1}{2} x_1^{(h)}(X)\, x_1^{(k)}(X)\, z_{0,x^{(h)}x^{(k)}}(X), \end{aligned}\right\} \tag{1.4}$$

and so on. Observe that (1.4) can be also obtained directly by introducing (1.2) in every term of the development (1.1) and expanding each term in Taylor series.

Equations (1.4) are all that is needed, in addition to the formal solution (1.1) of the untransformed equations with the corresponding b.c., in order to apply the coordinate perturbation technique, and the sometimes discouraging complication of the transformed equations can be entirely bypassed.

Indeed, once the sequence $z_0(x)$, $z_1(x)$, $z_2(x)$, ... is known, equations (1.4) are sufficient to indicate, often upon simple inspection, the suitable form of the straining coefficients $x_1^{(h)}$, $x_2^{(h)}$, ... and to determine the sequence $Z_0(X)$, $Z_1(X)$, $Z_2(X)$ appropriate for the uniform validity of the development (1.3), without any necessity of solving the transformed equations. It must be added that the straining coefficients contain an element of arbitrariness which contributes to the confusion when solving the transformed equations, but does not create any particular difficulty when making use of (1.4).

Pritulo's formulation can be easily applied no matter how large is the number of variables. It also holds, of course, when there is only one unknown and one independent variable, in which case the equation is simply an ordinary differential equation and the technique degenerates into the classical Poincaré technique. Indeed, also the application of the latter is substantially simplified by Pritulo's formulation, as shown by the simple illustrative example of § 2. It is remarkable that in so many years this simple formulation of Poincaré's method had never been pointed out.

We observe that the simplicity of the formulation also allows an easy and fast answer to the question whether the coordinate perturbation method will or will not apply to a particular problem.

In §§ 3 to 5 we present examples in the field of fluid mechanics, clearly showing how the procedure can be applied, to an order where the standard procedure would be unwieldy. They also show how drastically simpler the successive terms of the resulting expansions are when compared to those of a straightforward expansion. But, most important, it will become clear that in the field of fluid mechanics and in the presence of shocks the coordinate perturbation technique is incomplete unless it is supplemented by the multiple scale technique.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

2. Oscillator with nonlinear restoring force

This is a classical example for the Poincaré technique (see, for example, Cole (1968). The non-dimensional equation between the normalized displacement z and time t is

$$z_{tt}+z-\epsilon z^3 = 0, \tag{2.1}$$

with ϵ, small, representing a measure of the nonlinearity of the restoring force. Assume $z(0) = 1$, $z_t(0) = 0$, and expand as

$$z(t;\epsilon) = z_0(t)+\epsilon z_1(t)+\epsilon^2 z_2(t)+\dots, \tag{2.2}$$

so that (2.1) and the b.c. split into

$$\begin{aligned} z_{0tt}+z_0 &= 0; \qquad z_0(0) = 1, \quad z_{0t}(0) = 0; \\ z_{1tt}+z_1-z_0^3 &= 0; \qquad z_1(0) = 0, \quad z_{1t}(0) = 0; \\ z_{2tt}+z_2-3z_0^2 z_1 &= 0; \qquad z_2(0) = 0, \quad z_{2t}(0) = 0; \end{aligned}$$

and so on. Integrating we obtain

$$\begin{aligned} z_0(t) &= \cos t, \\ z_1(t) &= -\tfrac{1}{32}(\cos 3t-\cos t)+\tfrac{3}{8}t\sin t, \\ z_2(t) &= \tfrac{1}{1024}(\cos 5t-\cos t)-\tfrac{3}{128}(\cos 3t-\cos t)-\tfrac{9}{256}t\sin 3t+\tfrac{3}{32}t\sin t-\tfrac{9}{128}t^2\cos t. \end{aligned}$$

Evidently the secular terms in z_1 and z_2 prevent the uniform validity of the expansions (2.2), unless ϵt is small. However, if we write

$$z(t;\epsilon) = Z(T;\ \epsilon) = Z_0(T)+\epsilon Z_1(T)+\epsilon^2 Z_2(T)+\dots$$

with
$$t = T+\epsilon t_1(T)+\epsilon^2 t_2(T)+\dots$$

we immediately get from the application of (1.4)

$$\begin{aligned} Z_0(T) &= z_0(T) = \cos T, \\ Z_1(T) &= z_1(T)+t_1 z_{0t}(T) = -\tfrac{1}{32}(\cos 3T-\cos T)+[\tfrac{3}{8}T-t_1]\sin T, \\ Z_2(T) &= z_2(T)+t_1 z_{1t}(T)+t_2 z_{0t}(T)+\tfrac{1}{2}t_1^2 z_{0tt}(T) \\ &= \tfrac{1}{1024}(\cos 5T-\cos T)-\tfrac{3}{128}(\cos 3T-\cos T)-\tfrac{9}{256}T\sin 3T+\tfrac{3}{32}T\sin T \\ &\quad -\tfrac{9}{128}T^2\cos T+t_1[\tfrac{3}{32}\sin 3T+\tfrac{11}{32}\sin T+\tfrac{3}{8}T\cos T] \\ &\quad -t_2\sin T-\tfrac{1}{2}t_1^2\cos T. \end{aligned}$$

From the second equation it is immediately seen that the choice

$$t_1 = \tfrac{3}{8}T \tag{2.3}$$

eliminates from Z_1 the secular terms. Actually any bounded function could be added to (2.3), but obviously (2.3) represents the simplest choice. Similarly, after inserting this value of t_1, the third equation becomes

$$Z_2(T) = \tfrac{1}{1024}(\cos 5T-\cos T)-\tfrac{3}{128}(\cos 3T-\cos T)+[\tfrac{57}{256}T-t_2]\sin T,$$

so that the choice
$$t_2 = \tfrac{57}{256}T$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

eliminates from Z_2 the secular terms. Again, this value of t_2 represents obviously the simplest choice. Hence finally the solution of (2.1) is implicitly given by

$$z = \cos T - \tfrac{1}{32}(\cos 3T - \cos T)\epsilon + [\tfrac{1}{1024}(\cos 5T - \cos T) - \tfrac{3}{128}(\cos 3T - \cos T)]\epsilon^2 + \dots$$

with
$$t = [1 + \tfrac{3}{8}\epsilon + \tfrac{57}{256}\epsilon^2]T + \dots.$$

3. Supersonic aerofoil theory

The aerofoil sections considered here are sharply pointed at both their leading and trailing edges, and the flow deflexions are assumed to be so limited that no subsonic flow regions appear. This is a classical example, and has been widely treated with more or less rigour by Lighthill (1954), Ackeret (1925), Buseman (1935), Donov (1939), Friedrichs (1948), Van Dyke (1964) Legras (1953) and by a number of other authors. Here our purpose being a test of the mathematical method, we shall confine our attention only to the solution between the front and the rear shocks and to the determination of the front shock shape. It is expected that the application to the rest of the flow field can be conducted without additional difficulties along the same lines.

Since we are going to push the approximation to the third order, it is preferable to work on the original conservation equations without making simplifying statements about entropy or vorticity. Also, it will be interesting to obtain the results from the equations themselves, without the help of physical reasoning (so extensively used, for instance, by Lighthill 1954).

The flow equations for a perfect, polytropic and inviscid gas are

$$\nabla.(\rho \boldsymbol{q}) = 0; \quad M^2(\boldsymbol{q}.\nabla)\boldsymbol{q} + (1/\gamma\rho)\nabla p = 0; \quad \boldsymbol{q}.\nabla\sigma = 0.$$

Here the pressure p, density ρ and velocity $\boldsymbol{q}$ are normalized dividing by their undisturbed values (at infinity upstream), M is the Mach number in the undisturbed flow and σ is the entropy variation from the undisturbed condition, normalized dividing by the specific heat, so that we can write

$$\rho = p^{1/\gamma}\,\mathrm{e}^{-\sigma}. \tag{3.1}$$

This equation is used to eliminate the density from the above equations. The two-dimensional result, in scalar form, is

$$(1/\gamma)(up_x + vp_y) + p(u_x + v_y) = 0, \tag{3.2}$$

$$M^2(uu_x + vu_y) + (1/\gamma)p^{-1/\gamma}\,\mathrm{e}^{\sigma}p_x = 0, \tag{3.3}$$

$$M^2(uv_x + vv_y) + (1/\gamma)p^{-1/\gamma}\,\mathrm{e}^{\sigma}p_y = 0, \tag{3.4}$$

$$u\sigma_x + v\sigma_y = 0. \tag{3.5}$$

The boundary conditions are

$$p = 1, \quad u = 1, \quad v = 0, \quad \sigma = 0 \quad \text{at} \quad x = -\infty, \tag{3.6}$$

$$v/u = \epsilon W'(x) \quad \text{along} \quad y = \epsilon W(x), \tag{3.7}$$

where ϵ represents a wing thickness parameter and $W(x)$ defines the wing profile family. The prime indicates differentiation.

Along the shocks, the shape of which is to be determined, the conservation equations have to be satisfied. Indicating with Δf the jump of a quantity f at the shock, the conditions can be written in the form

$$\frac{\mathrm{d}y_{\mathrm{sh}}}{\mathrm{d}x} = \frac{\Delta(\rho v)}{\Delta(\rho u)} = -\frac{\Delta u}{\Delta v}, \tag{3.8}$$

$$\Delta\left[\frac{p}{\rho}+\frac{\gamma-1}{2}M^2(u^2+v^2)\right] = 0, \tag{3.9}$$

$$\Delta\left(\frac{1}{p}\right)\Delta\left(\frac{p}{\rho}\right)-\frac{\gamma-1}{2\gamma}\Delta p\,\Delta\left(\frac{1}{\rho^2}\right) = 0. \tag{3.10}$$

Equations (3.8) express the conservation of mass and of tangential momentum, (3.9) the conservation of energy and (3.10) corresponds to the Rankine–Hugoniot relation and follows from the conservation of normal momentum.

In a straightforward expansion procedure we introduce the expansions

$$\left.\begin{aligned} p &= 1+\epsilon p_1+\epsilon^2 p_2+\epsilon^3 p_3+\dots, & \sigma &= \epsilon\sigma_1+\epsilon^2\sigma_2+\epsilon^3\sigma_3+\dots,\\ u &= 1+\epsilon u_1+\epsilon^2 u_2+\epsilon^3 u_3+\dots, & v &= \epsilon v_1+\epsilon^2 v_2+\epsilon^3 v_3+\dots,\end{aligned}\right\} \tag{3.11}$$

in (3.2) to (3.5); as well as in the boundary and shock conditions (3.6) to (3.10), and separate the various orders.

Up to third order we obtain

$$\left.\begin{aligned} (1/\gamma)p_{1x}+u_{1x}+v_{1y} &= 0,\\ (1/\gamma)\,(p_{2x}+u_1p_{1x}+v_1p_{1y})+u_{2x}+v_{2y}+p_1(u_{1x}+v_{1y}) &= 0,\\ (1/\gamma)\,(p_{3x}+u_1p_{2x}+v_1p_{2y}+u_2p_{1x}+v_2p_{1y})+u_{3x}+v_{3y}+p_1(u_{2x}+v_{2y})+p_2(u_{1x}+v_{1y}) &= 0;\end{aligned}\right\} \tag{3.12}$$

$$\left.\begin{aligned} M^2u_{1x}+(1/\gamma)p_{1x} &= 0,\\ M^2(u_{2x}+u_1u_{1x}+v_1u_{1y})+(1/\gamma)p_{2x}-(1/\gamma)^2p_1p_{1x} &= 0,\\ M^2(u_{3x}+u_1u_{2x}+v_1u_{2y}+u_2u_{1x}+v_2u_{1y})+(1/\gamma)p_{3x}-(1/\gamma)^2(p_1p_{2x}+p_2p_{1x})+[(\gamma+1)/2\gamma^3]p_1^2p_{1x} &= 0;\end{aligned}\right\} \tag{3.13}$$

$$\left.\begin{aligned} M^2v_{1x}+(1/\gamma)p_{1y} &= 0,\\ M^2(v_{2x}+u_1v_{1x}+v_1v_{1y})+(1/\gamma)p_{2y}-(1/\gamma)^2p_1p_{1y} &= 0,\\ M^2(v_{3x}+u_1v_{2x}+v_1v_{2y}+u_2v_{1x}+v_2v_{1y})+(1/\gamma)p_{3y}-(1/\gamma)^2(p_1p_{2y}+p_2p_{1y})+[(\gamma+1)/2\gamma^3]p_1^2p_{1y} &= 0;\end{aligned}\right\} \tag{3.14}$$

$$\left.\begin{aligned} \sigma_{1x} &= 0,\\ \sigma_{2x}+u_1\sigma_1+v_1\sigma_{1y} &= 0,\\ \sigma_{3x}+u_1\sigma_{2x}+v_1\sigma_{2y}+u_2\sigma_{1x}+v_2\sigma_{1y} &= 0.\end{aligned}\right\} \tag{3.15}$$

Observe that in (3.13) and (3.14) there should appear terms containing σ_1 and σ_2. Since, however, these are later found to vanish throughout the field, the corresponding terms have been suppressed for simplicity. The b.c. (3.6) becomes simply

$$p_1 = u_1 = v_1 = \sigma_1 = p_2 = u_2 = v_2 = \sigma_2 = p_3 = u_3 = v_3 = \sigma_3 = 0 \quad \text{at} \quad x = -\infty. \tag{3.16}$$

The b.c. (3.7) at the aerofoil surface are transferred to $y = 0$ by using the expression

$$v(x,\epsilon W) = \epsilon v_1(x,0)+\epsilon^2[v_2(x,0)+W(x)v_{1y}(x,0)] + \epsilon^3[v_3(x,0)+W(x)v_{2y}(x,0)+\tfrac{1}{2}W^2(x)v_{1yy}(x,0)]+\dots$$

and similarly for $u(x,\epsilon W)$. One obtains

$$\left.\begin{aligned} v_1(x,0) &= W'(x),\\ v_2(x,0) &= W'(x)u_1(x,0)-W(x)v_{1y}(x,0),\\ v_3(x,0) &= W'(x)\,[u_2(x,0)+W(x)u_{1y}(x,0)]-W(x)v_{2y}(x,0)-\tfrac{1}{2}W^2(x)v_{1yy}(x,0).\end{aligned}\right\} \tag{3.17}$$

For the shock conditions we will write for the moment only those corresponding to the last equation (3.8), that is $\Delta u \Delta(\rho u) + \Delta v \Delta(\rho v) = 0$, or

$$\left.\begin{aligned} (\Delta u_1)^2 + (\Delta v_1)^2 + \Delta u_1 \Delta \rho_1 &= 0, \\ 2(\Delta u_1 \Delta u_2 + \Delta v_1 \Delta v_2) + \Delta u_1 \Delta \rho_2 + \Delta u_2 \Delta \rho_1 + \Delta u_1 \Delta(\rho_1 u_1) + \Delta v_1 \Delta(\rho_1 v_1) &= 0, \\ 2(\Delta u_1 \Delta u_3 + \Delta v_1 \Delta v_3) + (\Delta u_2)^2 + (\Delta v_2)^2 + \Delta u_1 \Delta(\rho_2 u_1 + \rho_1 u_2) + \Delta v_1 \Delta(\rho_2 v_1 + \rho_1 v_2) & \\ + \Delta u_2 \Delta(\rho_1 u_1) + \Delta v_2 \Delta(\rho_1 v_1) + \Delta u_2 \Delta \rho_2 + \Delta u_1 \Delta \rho_3 &= 0 \end{aligned}\right\} \tag{3.18}$$

and to (3.9) and (3.10), that is

$$\left.\begin{aligned} (1/\gamma)\Delta p_1 + M^2 \Delta u_1 &= 0, \\ (1/\gamma)\Delta p_2 - (1/2\gamma^2)\Delta p_1^2 + M^2 \Delta[u_2 + \tfrac{1}{2}(u_1^2 + v_1^2)] &= 0, \\ (1/\gamma)\Delta p_3 - (1/\gamma^2)\Delta(p_1 p_2) + [(\gamma+1)/6\gamma^3]\Delta p_1^3 + [(\gamma+1)/12\gamma^3](\Delta p_1)^3 + M^2 \Delta(u_3 + u_1 u_2 + v_1 v_2) &= 0; \end{aligned}\right\} \tag{3.19}$$

$$\left.\begin{aligned} \Delta \rho_1 - (1/\gamma)\Delta p_1 &= 0, \\ \Delta \rho_2 - (1/\gamma)\Delta p_2 + [(\gamma-1)/2\gamma^2]\Delta p_1^2 &= 0, \\ \Delta \rho_3 - (1/\gamma)\Delta p_3 + [(\gamma-1)/\gamma^2]\Delta(p_1 p_2) - [(\gamma-1)(2\gamma-1)/6\gamma^3]\Delta p_1^3 + [(\gamma^2-1)/12\gamma^3](\Delta p_1)^3 &= 0. \end{aligned}\right\} \tag{3.20}$$

Observe that the equations obtained from (3.10) have been manipulated so as to obtain the form (3.20), explicit in the $\Delta \rho_i$, which can immediately be compared to that obtained from the expansion of (3.1), showing that

$$\Delta \sigma_1 = \Delta \sigma_2 = 0; \quad \Delta \sigma_3 = [(\gamma^2-1)/12\gamma^3](\Delta p_1)^3. \tag{3.21}$$

This is the classical result that the entropy jump across a shock is of third order in the shock strength. The consequence of (3.21), (3.15) and (3.16) is that $\sigma_1 = \sigma_2 = 0$ throughout the flow field, in accordance with the aforesaid simplification of (3.13) and (3.14). Then (3.15) shows that σ_3 is independent of x between shocks, $\sigma_3(y)$ only changing at each shock by the amount (3.21).

The general solution of the first-order equations is

$$\begin{aligned} (1/\gamma)p_1 &= (M^2/B)[\phi_1(r) - \psi_1(s)], \\ u_1 &= -(1/B)[\phi_1(r) - \psi_1(s)] + \chi_1(y), \\ v_1 &= \phi_1(r) + \psi_1(s), \end{aligned}$$

where we have defined $\quad r = x - By, \quad s = x + By, \quad B^2 = M^2 - 1,$ (3.22)

and ϕ_1, ψ_1, χ_1 are arbitrary functions of the respective arguments. We see that, through (3.22), p_1, u_1 and v_1 can be considered functions of x, y. Alternatively, they may be considered functions of r, s since we have from (3.22)

$$y = (1/2B)(s - r). \tag{3.23}$$

Evidently r and s represent the characteristic coordinates of the linearized equations of flow, and ϕ_1 and ψ_1 represent outgoing and incoming waves, while χ_1 (constant along the third characteristic of the first-order equations) is related to the vorticity by $u_{1y} - v_{1x} = \chi_1'(y)$.

Application of the conditions (3.16) to (3.20) provides for the arbitrary functions the values

$$\phi_1(r) = W'(r), \quad \psi_1(s) = 0, \quad \chi_1(y) = 0.$$

Hence the first-order solution is irrotational and given by

$$(B/\gamma M^2)p_1 = -Bu_1 = v_1 = W'(r), \tag{3.24}$$

which represents the classical linear solution. That this solution satisfies the shock condition is immediately checked by calculating from it the jumps of the various quantities through any shock, on the two sides of which W' takes different values, W'_+ and W'_-, so that

$$\Delta W'_{sh} = W'_+ - W'_-.$$

We get
$$\Delta v_1 = -B\Delta u_1; \quad \Delta \rho_1 = (1/\gamma)\Delta p_1 = -M^2 \Delta u_1, \tag{3.25}$$

$$\Delta u_1 = -(1/B)\Delta W'_{sh}. \tag{3.26}$$

Equations (3.25) coincide indeed with the first-order shock conditions. In the present problem we have to take $\Delta W'_{sh} = W'_+$ if we concentrate on the front shock. Since in the linearized version its position should coincide with the characteristic $r = 0$ if $x = 0$ is the abscissa of the point where it crosses the x-axis, one would be tempted to take $W'_+ = W'(0)$, that is constant values for $\Delta W'_{sh}$ and all the other jumps. This evaluation, however, is only satisfactory for small values of ϵy, and fails to be uniformly valid at large values of this quantity. We will proceed to the correct evaluation later. For the moment we will only notice that the exact equation of the front shock is not $r_{sh} = 0$, but might be expressed, for instance, in the form

$$r_{sh} = G^*(s), \tag{3.27}$$

if the other characteristic coordinate is used as the argument, or in the form

$$r_{sh} = H^*(y) = G^*[s_{sh}(y)] \tag{3.28}$$

if the argument must be y. Here $s_{sh}(y)$ represents the one to one relation between s and y along the shock. Then
$$\Delta W'_{sh} = W'[G^*(s)] = W'[H^*(y)] \tag{3.29}$$

can be considered a function of s or a function of y.

Inserting (3.24) in the second-order equations (3.12) to (3.14) we obtain

$$(1/\gamma)p_{2x} + u_{2x} + v_{2y} = (\gamma+1)\,(M^4/B^2)\,W'(r)\,W''(r);$$
$$M^2 u_{2x} + (1/\gamma)p_{2x} = 0; \quad M^2 v_{2x} + (1/\gamma)p_{2y} = 0.$$

The general solution is

$$\begin{aligned} u_2 - \chi_2(y) &= -(1/\gamma M^2)p_2 \\ &= -(K_1/2B^2)\,[2ByW'(r)\,W''(r) + W'^2(r)] - (1/B)\,[\phi_2(r) - \psi_2(s)] \\ v_2 &= K_1 y W'(r)\,W''(r) + \phi_2(r) + \psi_2(s), \end{aligned}$$

where we have written for brevity

$$K_1(M) = (\gamma+1)M^4/2B^2.$$

Here ϕ_2, ψ_2 represent arbitrary outgoing and incoming second-order waves and χ_2 is related to an arbitrary second-order vorticity distribution by $u_{2y} - v_{2x} = \chi'_2(y)$.

Application of the conditions (3.16) to (3.20) provides for the arbitrary functions the values

$$\phi_2(r) = BW(r)\,W''(r) - (1/B)\,W'^2(r); \quad \psi_2(s) = 0; \quad \chi_2(y) = 0,$$

and the resulting second-order solution is also irrotational and given by

$$\left.\begin{aligned} u_2 &= -(1/\gamma M^2)p_2 = -(K_1/B)yW'(r)\,W''(r) - W(r)\,W''(r) + K_2\,W'^2(r), \\ v_2 &= K_1 y W'(r)\,W''(r) + BW(r)\,W''(r) - (1/B)\,W'^2(r), \end{aligned}\right\} \tag{3.30}$$

with $$K_2(M) = (1/B^2)\,(1-\tfrac{1}{2}K_1).$$

Again one can calculate from (3.30) with the help of (3.24) and (3.25) the following relations between the jumps of the various quantities at any shock:

$$\left.\begin{aligned}\Delta v_2 &= -B\,\Delta u_2 - \tfrac{1}{2}BK_1\,\Delta(u_1^2)\\ \Delta\rho_2+\tfrac{1}{2}(\gamma-1)M^4\Delta(u_1^2) &= (1/\gamma)\Delta p_2 = -M^2\Delta u_2,\end{aligned}\right\} \tag{3.31}$$

where, of course, $\Delta(u_1^2)$ is given by $(1/B^2)\Delta(W'^2_{\text{sh}})$ and Δu_2 should be obtained from (3.30). It can be checked by direct substitutions that (3.31) satisfy to the second-order shock conditions.

After insertion of the first- and second-order solutions (3.24) and (3.30) and some manipulations, the third-order equations (3.12) to (3.14) become

$$(1/\gamma)p_{3x}+u_{3x}+v_{3y} = \Phi_x,$$

$$M^2u_{3x}+(1/\gamma)p_{3x} = \Psi_x,$$

$$M^2v_{3x}+(1/\gamma)p_{3y} = \Psi_y,$$

with $$\Phi = (\gamma+1)M^4[(K_1/B^2)yW'^2(r)W''(r)+(1/B)W(r)W'(r)W''(r)-K_3W'^3(r)],$$

$$\Psi = (K_1M^2/6B)W'^3(r),$$

and $$K_3(M) = (1/B^3)\,[1-\tfrac{1}{2}K_1+\tfrac{1}{12}(3+4\gamma)M^2].$$

The general solution is

$$\left.\begin{aligned}u_3-\chi_3(y) &= -(1/\gamma M^2)p_3+(K_1/6B)W'^3(r)\\ &= -(K_1/B^3)\,\{\mathit{\Delta}(r)+By\mathit{\Delta}'(r)+\tfrac{1}{6}B^2K_1y^2[W'^3(r)]''\}-(1/B)\,[\phi_3(r)-\psi_3(s)],\\ v_3 &= (K_1/B^2)\,\{By\Xi'(r)+\tfrac{1}{6}B^2K_1y^2[W'^3(r)]''\}+\phi_3(r)+\psi_3(s)\end{aligned}\right\} \tag{3.32}$$

with $$\mathit{\Delta}(r) = B^2W(r)W'(r)W''(r)-K_4W'^3(r),$$

$$\Xi(r) = B^2W(r)W'(r)W''(r)-K_5W'^3(r),$$

$$K_4(M) = 1+\tfrac{1}{3}(\gamma+1)M^2[1-(M^2/B^2)];\quad K_5(M) = 1+\tfrac{1}{3}(\gamma+1)M^2[1-(M^2/2B^2)].$$

In (3.32) ϕ_3 and ψ_3 represent arbitrary outgoing or incoming third-order waves and χ_3 is related to the arbitrary third-order vorticity distribution, since indeed we get from (3.32)

$$u_{3y}-v_{3x} = \chi_3'(y).$$

At this point we have to apply the third-order wall and shock conditions, and this time it is better to show the details of their application.

First we obtain from (3.32) and the third-order wall condition (3.17)

$$v_3(x,0) = \phi_3(x)+\psi_3(x) = F(x), \tag{3.33}$$

with

$$F(x) = (1/B^2)\,(1-\tfrac{1}{2}K_1)W'^3(x)+\tfrac{1}{2}B^2[W^2(x)W''(x)]'-2(1+\tfrac{1}{2}K_1)W(x)W'(x)W''(x).$$

Next we can derive from (3.32), after comparison with (3.24) and (3.30), the following relations:

$$\left.\begin{aligned}(1/\gamma)p_3+u_3 &= -\tfrac{1}{12}(\gamma+1)M^6u_1^3+M^2\chi_3(y),\\ v_3+Bu_3 &= -BK_1\{u_1u_2+[\tfrac{1}{3}(\gamma+1)M^2]\,[1-(M^2/4B^2)]u_1^3\}+2\psi_3(r)+B\chi_3(y).\end{aligned}\right\} \tag{3.34}$$

On the other hand, from the third-order shock equations (3.18) to (3.20) we obtain with the help of (3.25) and (3.31) the following jump relations across any shock:

$$(1/\gamma)\Delta p_3 + M^2\Delta u_3 = -\tfrac{1}{12}(\gamma+1)M^6[\Delta(u_1^3) - (\Delta u_1)^3],$$
$$\Delta v_3 + B\,\Delta u_3 = -BK_1\{\Delta(u_1u_2) + \tfrac{1}{3}(\gamma+1)M^2[1-(M^2/4B^2)]\,[\Delta(u_1^3) - \tfrac{1}{4}(\Delta u_1)^3]\}.$$

The expressions on the l.h.s of these equations must coincide with those obtained by taking the jump across the shock of the r.h.s. of (3.34). Hence we obtain, taking for Δu_1 its value (3.26)

$$\left.\begin{aligned}\Delta\chi_3(y) &= -(K_1/6B)\,(\Delta W'_{\text{sh}})^3 = \chi_3(y) = -(K_1/6B)\,W'^3[H^*(y)],\\ \Delta\psi_3(s) &= K_6(\Delta W'_{\text{sh}})^3 = \psi_3(s) = K_6\,W'^3[G^*(s)],\end{aligned}\right\} \tag{3.35}$$

with
$$K_6(M) = \tfrac{1}{12}K_1\{1-(K_1/M^2)\,[1-(M^2/4B^2)]\}. \tag{3.36}$$

The first two terms of each of (3.35) apply to any shock, but the last two terms apply only to the front shock, where (3.29) holds. Indeed, these are the values of $\chi_3(y)$ and $\psi_3(s)$ in the region between the front and rear shocks, since both functions vanish in the undisturbed flow.

To these two equations we can associate that obtained, using equations (3.21), from the fact that for $\sigma_1 = \sigma_2 = 0$ the last (3.15) says that σ_3 is a function of y alone, as already pointed out. We get

$$\left.\begin{aligned}\Delta\sigma_3(y) &= (\gamma-1)M^2(K_1/6B)\,(\Delta W'_{\text{sh}})^3\\ &= \sigma_3(y) = (\gamma-1)M^2(K_1/6B)\,W'^3[H^*(y)],\end{aligned}\right\} \tag{3.37}$$

where again the last two terms of the equation apply to the region between the two shocks. From the first parts of (3.35) and (3.37) we see that after any number of shocks $\sigma_3(y) = -(\gamma-1)M^2\chi_3(y)$, so that one gets

$$(\gamma-1)M^2(v_{3x} - u_{3y}) = \sigma_3'(y), \tag{3.38}$$

a result that could have been directly obtained from the application of the well-known vorticity theorem, which would have actually provided a speedier determination of $\chi_3(y)$.

Once $\psi_3(s)$ is known after the front shock, as given by the second part of (3.35), we obtain from (3.33)

$$\phi_3(r) = F(r) - K_6\{W'[G^*(r)]\}^3, \tag{3.39}$$

and the complete expressions for the third-order unknowns can be obtained from (3.32). We observe, as we should, the presence not only of third-order vorticity and entropy variations, but also, through $\psi_3(s)$, of third-order waves coming in from the front shock, and altering through reflexion the outgoing waves $\phi_3(r)$. However, for the actual determination of these third-order quantities, we must know the shock shape (3.27). Deferring for the moment its determination, we shall first concentrate on the application of the coordinate perturbation technique.

We observe that up to this point all the labour required was aimed at determining the straightforward expansion, and had nothing to do with the coordinate perturbation technique. Application of this technique is made necessary by the fact that our expansions contain secular terms in y and y^2 (and, of course, higher powers of y for higher order terms) that make it valid only for small values of ϵy, that is only for the near flow field. If we require also the determination of the far flow field we must get rid of those secular terms. So we introduce the perturbed coordinates X and Y such that

$$x = X + \epsilon x_1 + \epsilon^2 x_2 + \ldots, \quad y = Y + \epsilon y_1 + \epsilon^2 y_2 + \ldots, \tag{3.40}$$

and the perturbed characteristic coordinates

$$\left.\begin{aligned}R &= X - BY = r - \epsilon r_1 - \epsilon^2 r_2 - \ldots,\\ S &= X + BY = s - \epsilon s_1 - \epsilon^2 s_2 - \ldots,\end{aligned}\right\} \tag{3.41}$$

where of course we have

$$r_1 = x_1 - By_1; \quad s_1 = x_1 + By_1; \quad r_2 = x_2 - By_2; \quad s_2 = x_2 + By_2. \tag{3.42}$$

We also define the new expansions

$$\left.\begin{aligned} p &= 1+\epsilon P_1+\epsilon^2 P_2+\epsilon^3 P_3+\ldots, \quad & \sigma &= \epsilon\Sigma_1+\epsilon^2\Sigma_2+\epsilon^3\Sigma_3+\ldots; \\ u &= 1+\epsilon U_1+\epsilon^2 U_2+\epsilon^3 U_3+\ldots, \quad & v &= \epsilon V_1+\epsilon^2 V_2+\epsilon^3 V_3+\ldots. \end{aligned}\right\} \tag{3.43}$$

The coefficients of all the above expansions are considered to be functions of X and Y, or of R and S, or combinations thereof. Applications of the relations (1.4) to, say, the v expansion gives

$$V_1 = v_1(R), \tag{3.44}$$

$$V_2 = v_2(R, Y) + r_1 v_{1r}(R). \tag{3.45}$$

$$V_3 = v_3(R, S, Y) + r_1 v_{2r}(R, Y) + y_1 v_{2r}(R, Y) + r_2 v_{1r}(R) + \tfrac{1}{2} r_1^2 v_{1rr}(R), \tag{3.46}$$

where $v_1(r)$ and $v_2(r, y)$ are given by (3.24) and (3.30), and $v_3(r, s, y)$ by (3.32) after substitution of $\phi_3(r)$ and $\psi_3(s)$ from (3.35) and (3.39).

Equation (3.45) is explicitly written as

$$V_2 = K_1 YW'(R) W''(R) + BW(R) W''(R) - (1/B) W'^2(R) + r_1 W''(R).$$

We see that elimination of the secular term is achieved by taking $r_1 = -K_1 YW'(R)$. However, it is immediately noticed that a better choice is

$$r_1 = -K_1 YW'(R) - BW(R), \tag{3.47}$$

which makes W'' disappear from the expression for V_2, and hence eliminates another cause of non-uniform validity of the expansion, which appears for wing profiles presenting discontinuous slopes, and hence locally infinite values of $W''(R)$.

That (3.47) represents the correct choice for r_1 is confirmed when one goes to the third order (3.46). Indeed, thanks to this choice, a very substantial simplification takes place in the otherwise cumbersome expression obtained for V_3 after the explicit expressions for v_1, v_2 and v_3 are substituted. The secular term in Y^2 vanishes identically, and the remaining terms are reduced, after a good deal of cancelling, to the relatively simple expression

$$\begin{aligned} V_3 = (1/B^2)(1-\tfrac{1}{2}K_1) W'^3(R) - K_6\{W'^3[G_*(R)] - W'^3[G_*(S)]\} \\ + [r_2 + K_1 y_1 W'(R) - K_7 YW'^2(R) - K_1 W(R) W'(R)] W''(R), \end{aligned}$$

with $$K_7(M) = (K_1/B)\{1 + (\gamma+1) M^2[1 - (M^2/2B^2)]\}.$$

If the uniform validity of the expansion has to be insured even for discontinuous slopes, the bracketed factor of the term in W'' must identically vanish. Since we dispose here of two quantities, r_2 and y_1, this can be achieved in an infinite number of ways. However, it is clear at first sight that the simplest choice is

$$y_1 = W(R); \quad r_2 = K_7 YW'^2(R). \tag{3.48}$$

Hence, considering (3.42) and (3.47) one obtains the first-order straining coefficients as

$$x_1 = -K_1 YW'(R), \quad y_1 = W(R), \tag{3.49}$$

Similarly, to obtain the above value of r_2, one can take the second-order straining coefficients as

$$x_2 = K_7 Y W'^2(R), \quad y_2 = 0. \tag{3.50}$$

It is easily verified that the lines R = const. and S = const. coincide respectively with the outgoing and the incoming characteristic lines of the original equations, the value of the corresponding constant being equal to the unperturbed abscissae of the points where the characteristic lines intersect the wing surface. We see also from (3.40) that, at least up to second order, the equation $y = \epsilon W(x)$ of the wing, when expressed in perturbed coordinates, is $Y = 0$, with $X = x$. So that the $Y - y$ correction of (3.40) merely represents a vertical translation following the profile.

The fact must be stressed here that, for simplest results, it is convenient to perturb both coordinates, contrary to Lighthill's original suggestion, followed also by Pritulo. Hence this may be so even when there are only waves travelling in one direction, and not exclusively when waves travel in both directions as in the cases treated by Lin (1954) with his characteristic coordinates approach, where both coordinates were perturbed. Indeed Van Dyke (1964), perturbing only one coordinate, obtained the result (3.47) for the single straining coefficient r_1, but had he gone to third order without the introduction of the other straining coefficient y_1 he would have needed a two-termed r_2, to be compared with the simpler expression (3.48). The complication is likely to increase with the order of the approximation.

We observe that all of the above formulation remains valid when the profile has a discontinuous slope. If the abscissa of the angular point is x^*, the slope will change abruptly there from a value $W'(x^*-)$ to a value $W'(x^*+)$. It must be clear then that at $R = x^*$, $W'(R)$ can take all values between those two. To any such value of W' correspond different values of the x_i, equations (3.49) and (3.50), and of the U_i, V_i, P_i, equation (3.51). These will furnish the position of, and the flow conditions on, different characteristic lines, all issuing from the angular point, each corresponding to a particular value of W'. Hence the 'expansion fan' produced by the angular point is properly represented.

Finally, the coefficients of the uniformly valid expansions (3.42) can be calculated from

$$\left.\begin{aligned}
V_1 &= -BU_1 = (B/\gamma M^2)\,P_1 = W'(R),\\
V_2 &= -(1/B)\,W'(R),\\
U_2 &= (1/\gamma M^2)\,P_2 = (1/B^2)\,(1-\tfrac{1}{2}K_1)\,W'^2(R),\\
V_3 &= (1/B^2)\,(1-\tfrac{1}{2}K_1)\,W'^3(R) - K_6\{W'^3[G^*(R)] - W'^3[G^*(S)]\},\\
U_3 + (K_1/6B)\,W'^3[H^*(Y)] &= -(1/\gamma M^2)\,P_3 + (K_1/6B)\,W'^3(R)\\
&= -(1/B^3)\,K_8\,W'^3(R) + (K_6/B)\,\{W'^3[G^*(R)] + W'^3[G^*(S)]\},\\
\Sigma_1 &= \Sigma_2 = 0, \quad \Sigma_3 = (\gamma-1)\,M^2(K_1/6B)\,W'^3[H^*(Y)],
\end{aligned}\right\} \tag{3.51}$$

with
$$K_8(M) = 1 - \tfrac{3}{2}K_1\{1 - \tfrac{2}{9}(\gamma+1)\,(M^2/B^2)\}.$$

Hence, as expected, the secular terms, as well as the terms in the higher derivatives of W, have disappeared also from the coefficients of the u and p expansions. Naturally, up to second order the expressions (3.51) coincide with well-known results. (See for instance Van Dyke 1964.)

We come now to the determination of the shock shape functions $H^*(y)$ and $G^*(s)$. It may be observed that, for the purpose of their use in (3.51), the shock shape needs only to be determined to the first order. This means that if (3.28), for instance, expressed in the perturbed coordinates, were expanded as

$$R_{sh} = H_0^*(Y) + \epsilon H_1^*(Y) + \dots \tag{3.52}$$

we would need only the function $H_0^*(Y)$, and similarly from $G_0^*(S)$. However, since we have already in our hands the elements for the determination of H_1^* and G_1^*, we will derive the shock shape including the first-order terms.

The derivation is based on the use of the first shock condition (3.8), still unused to this point. Let us write this condition in the form

$$\mathrm{d}r\,\Delta v - \mathrm{d}x\,\Delta(v+Bu) = 0, \tag{3.53}$$

all quantities being calculated at the shock, and replace in it, according to (3.41), (3.43), (3.47), (3.48) and (3.51):

$$\left.\begin{aligned} \mathrm{d}r &= \mathrm{d}R\{1-\epsilon[BW'(R)+K_1 YW''(R)]+2\epsilon^2K_7 YW'(R)W''(R)\} \\ &\quad +\mathrm{d}Y[-\epsilon K_1 W'(R)+\epsilon^2 K_7 W'^2(R)], \\ \mathrm{d}x &= \mathrm{d}R[1-\epsilon K_1 YW''(R)+2\epsilon^2 K_7 YW'(R)W''(R)] \\ &\quad +\mathrm{d}Y[B-\epsilon K_1 W'(R)+\epsilon^2 K_7 W'^2(R)], \\ \Delta v &= \epsilon W'(R)-\epsilon^2(1/B)W'^2(R), \\ \Delta(v+Bu) &= -\epsilon^2(K_1/2B)W'^2(R)+\epsilon^3 K_9 W'^3(R), \end{aligned}\right\} \tag{3.54}$$

with $$K_9(M) = (K_1/B^2)\{1+[(\gamma+1)/4]M^2[1-(5M^2/4B^2)]\},$$

and where we have written the developments up to the necessary order only. In view of (3.52), we can also replace in the above equations the developments

$$W'(R) = W'(H_0^*)+\epsilon H_1^* W''(H_0^*)+\ldots; \quad W''(R) = W''(H_0^*)+\epsilon H_1^* W'''(H_0^*)+\ldots.$$

The result of the expansion of (3.53) is that $H_0^*(Y) = \text{const.}$ ($=0$ if the front shock goes through the origin) and $H_1^*(Y) = (K_1/2)W'(H_0^*)Y$. Similarly, $H_2^*(Y)$ would contain terms in Y^2 and so on. As a result the expansion (3.52) is only valid for small ϵY and is not uniformly valid in the whole field of flow. Incidentally, the inadequacy of this expansion is confirmed by the fact that the third-order vorticity and entropy appearing in (3.51) would be constant throughout the field, while on physical grounds we know they should vanish at large Y.

To restore the uniform validity of the expansion (3.52) it is necessary to introduce a second scale for the coordinate Y,† by picking for the description of the shock shape the additional transverse variable

$$\eta = \epsilon Y. \tag{3.55}$$

Moreover, it is convenient to consider for the moment η_{sh} as a function of R (along the shock), rather than R_{sh} as a function of η and introduce the development

$$\eta_{sh}(R) = \eta_0(R)+\epsilon\eta_1(R)+\ldots. \tag{3.56}$$

With these changes the first two (3.54) become

$$\begin{aligned} \mathrm{d}r &= -K_1 W'\mathrm{d}\eta_0+(1-K_1\eta_0 W'')\mathrm{d}R \\ &\quad +\epsilon[-K_1 W'\mathrm{d}\eta_1+K_7 W'^2\mathrm{d}\eta_0-(BW'+K_1\eta_1 W''-2K_7\eta_0 W'W'')\,\mathrm{d}R], \\ \epsilon\,\mathrm{d}x &= B\,\mathrm{d}\eta_0+\epsilon[B\,\mathrm{d}\eta_1-K_1 W'\mathrm{d}\eta_0+(1-K_1\eta_0 W'')\,\mathrm{d}R], \end{aligned}$$

where again only the necessary terms have been written, and the W', W'' stand for $W'(R)$, $W''(R)$.

† In a private communication to the author, S. H. Lam observes that the need to switch from x to $\xi = \epsilon x$ arises every time the coefficients $z_n(x)$ of the expansion (1.1) for $z(\epsilon, x)$ are polynomials of order n. This is indeed what happens in the case of the development (3.52), and can be used as one general rule for the introduction of the new scale.

Equation (3.53) splits now in the following equations:

$$\tfrac{1}{2}K_1 W'(\mathrm{d}\eta_0/\mathrm{d}R) + K_1 W''\eta_0 = 1,$$

$$\tfrac{1}{2}K_1 W'(\mathrm{d}\eta_1/\mathrm{d}R) + K_1 W''\eta_1 = K_{10} W'^2(\mathrm{d}\eta_0/\mathrm{d}R) - K_{11} W' + K_{12} W'W''\eta_0,$$

with

$$K_{10} = (K_1/B)\,[1+\tfrac{3}{4}(\gamma+1)M^2-\tfrac{7}{8}K_1],$$

$$K_{11} = (1/B)\,[M^2-\tfrac{1}{2}K_1],$$

$$K_{12} = (K_1/B)\,[3+2(\gamma+1)M^2-\tfrac{5}{2}K_1].$$

From the first equation we get $\tfrac{1}{2}K_1\eta_0 = W(R)/W'^2(R)$.

Using this value of η_0, the second equation gives

$$(K_1/2)\eta_1 = K_{13}\,W(R)/W'(R) - K_{14}[1/W'^2(R)]\int_0^R W'^2(t)\,\mathrm{d}t,$$

with

$$K_{13}(M) = (2/B)\,\{1+\tfrac{1}{2}(\gamma+1)M^2-\tfrac{3}{4}K_1\},$$

$$K_{14}(M) = (\tfrac{1}{2}B)\,[(\gamma-1)M^2+\tfrac{1}{2}K_1].$$

In the determination of η_0 and η_1 the condition that $\eta = \eta_0 = \eta_1 = 0$ at $R = 0$ (where $W(0) = 0$) has been applied. One can easily verify Lighthill's result that, to first order, the shock bisects the angle between the perturbed and the unperturbed outgoing Mach waves.

The expansion (3.56) can be inverted to give R_{sh} in the form, similar to (3.52) except for the argument,

$$R_{\mathrm{sh}} = H_0(\eta) + \epsilon H_1(\eta) + \dots.$$

We find

$$H_0(\eta) = \Theta(\tfrac{1}{2}K_1\eta),$$

where the function Θ is obtained by inversion of W/W'^2, so that

$$\Theta^{-1}(t) = W(t)/W'^2(t),$$

after which we obtain

$$H_1(\eta) = \frac{-K_{13}\,W[H_0(\eta)]\,W'[H_0(\eta)] + K_{14}\displaystyle\int_0^{H_0(\eta)} W'^2(t)\,\mathrm{d}t}{W'[H_0(\eta)]\,\{1-2K_1\,W''[H_0(\eta)]\}}.$$

For the purpose of the third-order field calculation we need only $H_0(\eta)$, but we also need the zero-order shock shape in terms of S. This is immediately obtained from the fact that

$$\eta = (\epsilon/2B)\,(S-R).$$

Since $R_{\mathrm{sh}} = H_0(\eta)+\dots$ is always of order higher than S, we can write that along the shocks

$$\eta = (\tfrac{1}{2}B)\,\epsilon S = \zeta/2B,$$

where we have had to introduce a new scale, $\zeta = \epsilon S$, for S. Then, to zero order, we have along the shock

$$R_{\mathrm{sh}} = G_0(\zeta) = H_0(\zeta/2B) = \Theta(K_1\zeta/4B).$$

Finally, we get the complete third-order solution by taking in (3.51)

$$\left.\begin{aligned} W'^3[H^*(Y)] &= W'^3[\Theta(K_1\eta/2)],\\ W'^3[G^*(S)] &= W'^3[\Theta(K_1\zeta/4B)],\\ W'^3[G^*(R)] &= W'^3[\Theta(K_1\theta/4B)],\end{aligned}\right\} \tag{3.57}$$

where it has been necessary to introduce a second scale, $\theta = \epsilon R$, also for the R variable. We observe that the introduction of second scales was not preassumed. It was forced upon us by the uniform validity of the shock shape equation. Once their necessity is realized, however, the orthodox procedure (see, for instance, Cole 1968) would require us to start again from the original equations after introducing the second scales. It may be checked that the third-order results coincide with those obtained from our less orthodox procedure.

In order to illustrate the above developments, we apply these results to a wing of parabolic shape and unit cord, $W(x) = x(1-x)$. Then to zero order

$$(K_1/2)\eta_{\mathrm{sh}} = W(H_0)/W'^2(H_0) = \tfrac{1}{4}[(1-2H_0)^{-2}-1],$$

so that
$$R_{\mathrm{sh}} = H_0(\eta) = \tfrac{1}{2}[1-(1+2K_1\eta)^{-\frac{1}{2}}]$$

provides the zero-order shock shape, and

$$H_1(\eta) = -\tfrac{1}{2}K_{13}K_1\eta(1+2K_1\eta)^{-\frac{3}{2}} + \tfrac{1}{6}K_{14}(1+2K_1\eta)^{-\frac{1}{2}}[1-(1+2K_1\eta)^{-3}],$$

the first-order correction coefficient.

The jump across the shock is given by

$$\Delta W'_{\mathrm{sh}} = W'[H^*(Y)] = 1-2R_{\mathrm{sh}} = (1+2K_1\eta)^{-\frac{1}{2}} - 2\epsilon H_1(\eta) + \dots,$$

and steadily decreases with η, vanishing for $\eta \to \infty$ as it must indeed.

Finally, the quantities appearing in (3.57), necessary for the calculation of (3.51), become respectively

$$(1+2K_1\eta)^{-\frac{3}{2}}; \quad [1+(K_1\zeta/B)]^{-\frac{3}{2}}; \quad [1+(K_1\theta/B)]^{-\frac{3}{2}}.$$

It is interesting to observe that the above determination of the shock and of the flow field works also when the wing section contains concavities. As a simple example take $W(x) = x^2(1-x)^2$ for the region $0 \leqslant x \leqslant 1$, and zero for $x < 0$, corresponding to a profile which, if symmetric, presents cusps, rather than wedges, at both ends. One obtains to zero order

$$\tfrac{1}{2}K_1\eta_{\mathrm{sh}} = \tfrac{1}{4}(1-2H_0)^{-2}$$

and hence
$$R_{\mathrm{sh}} = H_0(\eta) = \tfrac{1}{2}[1-(2K_1\eta)^{-\frac{1}{2}}].$$

Thus to zero order R_{sh} is $\gtrless 0$ as $2K_1\eta \gtrless 1$. So the first-order jump across the front shock is

$$\Delta W'_{\mathrm{sh}} = W'[H^*(Y)] = \tfrac{1}{2}(2K_1\eta)^{-\frac{1}{2}}[1-(2K_1\eta)^{-1}]$$

for $2K_1\eta \geqslant 1$, and $\Delta W'_{\mathrm{sh}} = 0$ for $2K_1\eta < 1$. This corresponds to the shock being formed away from the leading edge at a zero-order vertical distance $Y_{\mathrm{min}} = (2K_1\epsilon)^{-1}$. Its strength vanishes at $Y = Y_{\mathrm{min}}$ and as $Y \to \infty$, and, again to zero-order, its maximum corresponds to the maximum jump level $(\Delta W'_{\mathrm{sh}})_{\mathrm{max}} = 3^{-\frac{3}{2}}$ at $3Y_{\mathrm{min}}$. The first-order corrections are derived without difficulty.

Based on the above relations between R_{sh} and η, and on the connexions between these and the original variables, it is an easy matter to derive, to various orders, the shock-shape equation in the physical plane.

Previously there have been several attempts to derive a shock-shape equation uniformly valid everywhere (see, for instance, Legras (1953)). The result has never been satisfactory except in the case of Friedrichs (1948) (see the exhaustive discussion of Lighthill (1954)), who has succeeded in providing a valid approximate parametric representation of the shock shape. The determination presented here, based on the use of second scales, has the advantages of resulting in simpler expressions and of providing a direct relation with the flow field, thus leading without difficulty to higher order approximations.

Obviously the above procedure can be applied to the determination of the rear shock shape, and of the field behind the rear shock, and to the determination of the complete flow field also on the downward side of the wing when the section is not symmetric or symmetrically located. The additional feature of this determination is that the values of $W(x)$ in the region following the wing (corresponding to the shape of the dividing streamline) are not known in advance and must be determined by matching the upper and lower flow fields. It is actually a known result (Lighthill 1954) that the values of $W(x)$ vanish to first- and second-order accuracy, but become different from zero in the third-order approximation (third-order downwash).

If still higher approximations to the flow field were required, it is now evident that a double scale is necessary for the perturbed characteristic coordinates, as well as for Y, and hence also for X, and the corresponding uniformly valid development, can only be obtained by supplementing the coordinate perturbation technique by the multiple scale technique.

4. Finite amplitude organ pipe oscillations

The one-dimensional unsteady flow equations of a perfect polytropic inviscid gas are

$$\left.\begin{aligned}(1/\gamma)p_t+(1/\gamma)up_x+pu_x &= 0,\\ \rho(u_t+uu_x)+(1/\gamma)p_x &= 0,\\ \sigma_t+i\sigma_x &= 0,\end{aligned}\right\} \tag{4.1}$$

where again

$$\rho = p^{1/\gamma}\,\mathrm{e}^{-\sigma}$$

has been used to transform the usual continuity equation to the form of the first (4.1). Here p and ρ have been normalized through their undisturbed values, u through the undisturbed speed of sound, x through a reference length (we choose the length of the pipe), and t through the corresponding undisturbed wave propagation time (reference length/undisturbed speed of sound).

It is useful to introduce also the normalized sonic velocity

$$a = p/\rho = p^{(\gamma-1)/2\gamma}\,\mathrm{e}^{\frac{1}{2}\sigma}. \tag{4.2}$$

The boundary conditions

$$u = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = 1 \tag{4.3}$$

are supplemented by the knowledge of the initial distribution of p, u and σ. Moreover, across any shock the shock conservation equations must be satisfied. Here, since we are limiting ourselves to the second order we can, instead of writing the conservation equations, make use of the known property that the so-called Riemann invariants $[a/(\gamma-1)]\pm u$ are – up to second order – constant across shocks moving respectively in the negative or positive x-direction (which we shall indicate respectively as s-shocks or r-shocks). Hence developing according to (4.2), and indicating with Δ the jump of any quantity through the shock, we have

$$\Delta[(p_1/\gamma)\pm u_1] = 0, \quad \Delta[(p_2/\gamma)\pm u_2] = [(\gamma+1)/4\gamma^2]\,\Delta(p_1^2); \tag{4.4}$$

the upper and lower signs holding respectively on s- or on r-shocks. In (4.4) use has already been made of the known property that across any shock

$$\Delta\sigma_1 = \Delta\sigma_2 = 0,$$

so that σ_1 and σ_2 remain zero if, as we shall assume, they vanish initially. Finally use can be made of the known property that, again up to first order, the shock velocity is equal to the mean of the wave velocities on both sides of the shock,

$$(\mathrm{d}x_{\mathrm{sh}}/\mathrm{d}t) = (\mp a+u)_m. \tag{4.5}$$

From now on, we shall indicate with f_m the arithmetic mean of the values of the quantity f on the two sides of a shock.

Let us take for the variables the expansions

$$\left.\begin{aligned} p &= 1+\epsilon p_1+\epsilon^2 p_2+\ldots; \quad u = \epsilon u_1+\epsilon^2 u_2+\ldots; \quad \sigma = \epsilon\sigma_1+\epsilon^2\sigma_2+\ldots, \\ \rho &= 1+\epsilon\rho_1+\epsilon^2\rho_2+\ldots; \quad a = 1+\epsilon a_1+\epsilon^2 a_2+\ldots, \end{aligned}\right\} \tag{4.6}$$

where ϵ represents a small parameter representative of the perturbation amplitude. Substituting in (4.1), we obtain the first-order perturbation equations

$$(1/\gamma)p_{1t}+u_{1x} = 0; \quad u_{1t}+(1/\gamma)p_{1x} = 0; \quad \sigma_{1t} = 0 \tag{4.7}$$

from which we obtain the well-known acoustic solution

$$(1/\gamma)p_1 = \rho_1 = \phi_1(r)+\psi_1(s); \quad u_1 = \phi_1(r)-\psi_1(s); \quad \sigma_1 = 0, \tag{4.8}$$

the last being so if the initial disturbance is homentropic, in which case the first equation provides also ρ_1. Here $r = t-x$ and $s = t+x$ are the linear characteristic coordinates, of which, respectively, ϕ_1 and ψ_1 are arbitrary functions.

The conditions $u = 0$ at $x = 0$ and 1, obtained from expanding the conditions (4.3), become

$$\phi_1(t) = \psi_1(t); \quad \phi_1(t-1) = \psi_1(t+1) = \phi_1(t+1), \tag{4.9}$$

that is ϕ_1 and ψ_1 are one and the same function, periodic with period 2. The values of $\phi_1(t)$ in the interval 0 to 2 can immediately be calculated from the known initial distribution of p_1 and u_1.

The solution (4.8) can be inserted in the second-order perturbation equations obtained from (4.1) by using the expansions (4.6), and these can in turn be integrated, with the result that secular terms in t appear in the solution. As a consequence for sufficiently large t the second-order terms of the expansions (4.6) grow larger than the first-order terms and the expansions fail to be uniformly valid. Physically, this is due to the fact that the nonlinear solution cannot be periodic, and that the initial wave shape undergoes a slow distortion, which only vanishes when $\epsilon \to 0$.

This effect, due to the lack of parallelism of the nonlinear characteristics, can be taken care of, as in the treatment of progressive waves, with the help of the multiple scale technique. We introduce a second time variable

$$\theta = \epsilon t, \tag{4.10}$$

and we substitute

$$\partial/\partial t+\epsilon\,\partial/\partial\theta$$

in lieu of $\partial/\partial t$ in (4.1), as if t and θ were independent variables. After this replacement and the substitution of the expansions (4.6), we obtain again the first-order perturbation equations in the form (4.7), but their solution will be this time

$$(1/\gamma)p_1 = \rho_1 = \phi_1(r,\theta)+\psi_1(s,\theta), \quad u_1 = \phi_1(r,\theta)-\psi_1(s,\theta), \quad \sigma_1 = 0, \tag{4.11}$$

where again application of the first-order conditions obtained from (4.3) gives

$$\phi_1(t,\theta) = \psi_1(t,\theta), \quad \phi_1(t-1,\theta) = \psi_1(t+1,\theta) = \phi_1(t+1,\theta), \tag{4.12}$$

showing that ϕ_1 and ψ_1 are one and the same function, periodic in t with period 2 (but still depending separately on θ). In spite of the identity of the two functions, in what follows we shall still write for brevity ϕ_1 to indicate $\phi_1(r,\theta)$ and ψ_1 to indicate $\phi_1(s,\theta)$.

The second-order perturbation equations, after substituting the first-order solution (4.11), become

$$(1/\gamma)p_{2t}+u_{2x}+\phi_{1\theta}+\psi_{1\theta}-(\phi_1-\psi_1)(\phi_{1r}-\psi_{1s})-\gamma(\phi_1+\psi_1)(\phi_{1r}+\psi_{1s}) = 0,$$
$$u_{2t}+(1/\gamma)p_{2x}+\phi_{1\theta}-\psi_{1\theta}-(\phi_1-\psi_1)(\phi_{1r}+\psi_{1s})+(\phi_1+\psi_1)(\phi_{1r}-\psi_{1s}) = 0,$$
$$\sigma_{2t} = 0.$$

Integration of the third equation gives $\sigma_2 = 0$ if the initial disturbance is homentropic. In view of the definition of r and s the first two equations can be combined as

$$\partial[(p_2/\gamma)+u_2]/\partial s = -\phi_{1\theta}+(\phi_1-\psi_1)\,\phi_{1r}+\tfrac{1}{2}(\phi_1+\psi_1)\,[(\gamma-1)\,\phi_{1r}+(\gamma+1)\,\psi_{1s}],$$
$$\partial[(p_2/\gamma)-u_2]/\partial r = -\psi_{1\theta}-(\phi_1-\psi_1)\,\psi_{1s}+\tfrac{1}{2}(\phi_1+\psi_1)\,[(\gamma+1)\,\phi_{1r}+(\gamma-1)\,\psi_{1s}].$$

We integrate the first with respect to s, keeping r constant. It is convenient to choose r as the arbitrary constant representing the lower limit of integration. Similarly, we integrate the second equation with respect to r choosing s as the lower integration limit.

The result is

$$\left.\begin{aligned}(p_2/\gamma)+u_2 &= [\tfrac{1}{2}(\gamma+1)\,\phi_1\phi_{1r}-\phi_{1\theta}]\,(s-r)-\tfrac{1}{2}(3-\gamma)\,\phi_{1r}\int_r^s\psi_1\,\mathrm{d}s' \\ &\quad +\tfrac{1}{2}(\gamma+1)\,\phi_1(\psi_1-\phi_1)+\tfrac{1}{4}(\gamma+1)\,(\psi_1^2-\phi_1^2)+2\phi_2^*, \\ (p_2/\gamma)-u_2 &= [\tfrac{1}{2}(\gamma+1)\,\psi_1\psi_{1s}-\psi_{1\theta}]\,(r-s)-\tfrac{1}{2}(3-\gamma)\,\psi_{1s}\int_s^r\phi_1\,\mathrm{d}r' \\ &\quad +\tfrac{1}{2}(\gamma+1)\,\psi_1(\phi_1-\psi_1)+\tfrac{1}{4}(\gamma+1)\,(\phi_1^2-\psi_1^2)+2\psi_2^*.\end{aligned}\right\} \quad (4.13)$$

Here $\phi_2^* = \phi_2^*(r,\theta)$ and $\psi_2^* = \psi_2^*(s,\theta)$ are arbitrary functions of the respective arguments and r' and s' are running variables.

It is appropriate at this point to introduce the coordinate perturbation defined by

$$x = X+\epsilon x_1+\ldots,\quad t = T+\epsilon t_1+\ldots \quad (4.14)$$

or, in terms of the characteristic coordinates, by

$$r = R+\epsilon r_1+\ldots,\quad s = S+\epsilon s_1+\ldots. \quad (4.15)$$

Here $\quad R = T-X,\quad S = T+X,\quad r_1 = t_1-x_1,\quad s_1 = t_1+x_1.$

We also introduce the expansions

$$\left.\begin{aligned}u &= \epsilon U_1+\epsilon^2 U_2+\ldots, & p &= 1+\epsilon P_1+\epsilon^2 P_2+\ldots, \\ \sigma &= \epsilon\Sigma_1+\epsilon^2\Sigma_2+\ldots, & a &= 1+\epsilon A_1+\epsilon^2 A_2+\ldots.\end{aligned}\right\} \quad (4.16)$$

In all of the above expansions the coefficients of the various powers of ϵ are to be considered functions of the perturbed coordinates and, of course, of θ.

Application of (1.4) gives the relation between the coefficients of the expansions (4.16) and (4.6). We obtain

$$\left.\begin{aligned}P_1 &= p_1(R,S,\theta) = \gamma[\phi_1(R,\theta)+\phi_1(S,\theta)], \\ U_1 &= u_1(R,S,\theta) = \phi_1(R,\theta)-\phi_1(S,\theta), \\ \Sigma_1 &= 0.\end{aligned}\right\} \quad (4.17)$$

and

$$\left.\begin{aligned}(1/\gamma)\,P_2+U_2 &= (1/\gamma)\,p_2(R,S,\theta)+u_2(R,S,\theta)+2\phi_{1r}(R,\theta)\,r_1(R,S,\theta), \\ (1/\gamma)\,P_2-U_2 &= (1/\gamma)\,p_2(R,S,\theta)-u_2(R,S,\theta)+2\phi_{1r}(S,\theta)\,s_1(R,S,\theta), \\ \Sigma_2 &= 0.\end{aligned}\right\} \quad (4.18)$$

The determination of r_1 and s_1 comes from the application of the conditions (4.5). If there are shocks, for the uniform validity of the expansions we may require the discontinuity of all the coefficients P_1, U_1, P_2, U_2, ... to take place at the same values of R and S. Hence the position of the shocks must correspond to constant values of R or S for r-shocks or s-shocks respectively. This means

$$(\mathrm{d}X/\mathrm{d}T)_{\mathrm{sh}} = \mp 1 \tag{4.19}$$

respectively along s- or r-shocks. In this case, and in the assumption that there is only one shock coming and going (fundamental mode), the T-period is always 2, and the frequency ω, in the perturbed time, remains constantly equal to π, so that we have always $\omega t = \pi T$. Hence if the expansion for ω is

$$\omega = \pi + \epsilon\omega_1 + \ldots, \tag{4.20}$$

comparing with (4.14) we obtain

$$\omega_1 = -\pi(t_1/T). \tag{4.21}$$

The following determination of T and X is carried out in the assumption that shock waves are present. In the opposite case when the waves are continuous obviously the period is not determined by the propagation time of the shock. The discussion will show how this case is handled.

From (4.14) we obtain

$$\mathrm{d}x = \mathrm{d}X\{1 + \epsilon[(x_{1R} + x_{1S})(\mathrm{d}T/\mathrm{d}X) + x_{1S} - x_{1R}] + \ldots\},$$

$$\mathrm{d}t = \mathrm{d}T\{1 + \epsilon[t_{1R} + t_{1S} + (t_{1S} - t_{1R})(\mathrm{d}X/\mathrm{d}T)] + \ldots\}.$$

Hence using (4.19) one finds for the s-shocks

$$(\mathrm{d}x/\mathrm{d}t)_{s\text{-sh}} = -1 + 2\epsilon(x_{1R} + t_{1R}) + \ldots = -1 + 2\epsilon s_{1R} + \ldots$$

and for the r-shocks

$$(\mathrm{d}x/\mathrm{d}t)_{r\text{-sh}} = 1 + 2\epsilon(x_{1S} - t_{1S}) + \ldots = 1 - 2\epsilon r_{1S} + \ldots.$$

Application of the condition (4.5) gives, after introduction of the expansions (4.16), with $A_1 = [(\gamma-1)/2\gamma]\,P_1$ by virtue of (4.2),

$$(\mathrm{d}x/\mathrm{d}t)_{\mathrm{sh}} = \mp 1 + \epsilon\{U_1 \mp [(\gamma-1)/2\gamma]\,P_1\}_m + \ldots$$

with U_1, P_1, given by (4.17). Taking into account the fact that along s-shocks or r-shocks only $\phi_1(S, \theta)$ or $\phi_1(R, \theta)$ respectively undergo a discontinuity, and comparing with the two preceding expansions for $(\mathrm{d}x/\mathrm{d}t)_{\mathrm{sh}}$ we get the two equations

$$r_{1S} = \tfrac{1}{4}(3-\gamma)\,\phi_1(S, \theta) - \tfrac{1}{4}(\gamma+1)\,\phi_{1m},$$

$$s_{1R} = \tfrac{1}{4}(3-\gamma)\,\phi_1(R, \theta) - \tfrac{1}{4}/(\gamma+1)\,\phi_{1m},$$

integrating we get

$$r_1 = \tfrac{1}{4}(3-\gamma)\int_R^S \phi_1(R', \theta)\,\mathrm{d}R' - \tfrac{1}{4}(\gamma+1)\,\phi_{1m}(S-R) + f_1(R),$$

$$s_1 = -\tfrac{1}{4}(3-\gamma)\int_R^S \phi_1(R', \theta)\,\mathrm{d}R' + \tfrac{1}{4}(\gamma+1)\,\phi_{1m}(S-R) + g_1(S),$$

where f_1 and g_1 are arbitrary functions of the respective arguments. Hence we obtain

$$2x_1 = s_1 - r_1 = -\tfrac{1}{2}(3-\gamma)\int_R^S \phi_1(R', \theta)\,\mathrm{d}R' + \tfrac{1}{2}(\gamma+1)\phi_{1m}(S-R) + g_1(S) - f_1(R),$$

$$2t_1 = s_1 + r_1 = g_1(S) + f_1(R).$$

It is convenient to choose f_1 and g_1 so that x_1 vanishes at $x = 0$ and 1, so that the ends of the tube correspond to $X = 0$ and 1. The simplest choice is

$$f_1(T) = g_1(T) = [\tfrac{1}{2}(3-\gamma)\langle\phi_1\rangle - \tfrac{1}{2}(\gamma+1)\,\phi_{1m}]\,T, \tag{4.22}$$

where
$$\langle\phi_1\rangle = \frac{1}{2}\int_T^{T+2} \phi_1(T',\theta)\,\mathrm{d}T'$$

represents the mean value of ϕ_1 over a cycle. Observe that both $\langle\phi_1\rangle$ and ϕ_{1m} can be functions of θ.

With this choice we get

$$\left.\begin{aligned} r_1 &= \tfrac{1}{4}(3-\gamma)\left[\int_R^S \phi_1(R',\theta)\,\mathrm{d}R' + 2R\langle\phi_1\rangle\right] - \tfrac{1}{4}(\gamma+1)\,\phi_{1m}(R+S),\\ s_1 &= \tfrac{1}{4}(3-\gamma)\left[-\int_R^S \phi_1(R',\theta)\,\mathrm{d}R' + 2S\langle\phi_1\rangle\right] - \tfrac{1}{4}(\gamma+1)\,\phi_{1m}(R+S),\end{aligned}\right\} \tag{4.23}$$

$$\left.\begin{aligned} x_1 &= -\tfrac{1}{4}(3-\gamma)\int_R^S [\phi_1(R',\theta) - \langle\phi_1\rangle]\,\mathrm{d}R',\\ t_1 &= [\tfrac{1}{2}(3-\gamma)\,\langle\phi_1\rangle - \tfrac{1}{2}(\gamma+1)\,\phi_{1m}]\,T.\end{aligned}\right\} \tag{4.24}$$

Again, we notice that both coordinates need being perturbed in order to obtain uniformly valid results.

Comparing with (4.21) we obtain

$$\omega_1 = \pi[\tfrac{1}{2}(\gamma+1)\,\phi_{1m} - \tfrac{1}{2}(3-\gamma)\,\langle\phi_1\rangle]. \tag{4.25}$$

It is interesting to observe that, contrary to what happens in the preceding example of the supersonic wing, the perturbed characteristic coordinates are constant not along the characteristic lines of the original equations, but along shocks. Alternatively, one could elect to make them constant along the characteristic lines and determine the shock shape in terms of the perturbed characteristic coordinates using a procedure similar to that of the preceding example, where actually the procedure was dictated by the need of suppressing secular terms. However, this procedure would be less rational in the present example, where the suppression of the secular terms already follows from the application of the condition of periodicity, so that we are free to use the coordinate perturbation to substantially simplify the determination of the shock.

The values (4.13) and (4.23) can now be used to calculate the expressions on the r.h.s. of (4.18), after which the values of P_2 and U_2 can immediately be obtained. The expression for P_2/γ contains a term

$$\{[\tfrac{1}{4}(3-\gamma)]\,\langle\phi_1\rangle - (\gamma+\tfrac{1}{4})\,\phi_{1m}\}\,(\phi_{1R}+\psi_{1S})\,(R+S),$$

which grows secularly with time. This secular growth, acceptable in the expressions (4.23) for R and S, and in the expression (4.24) for T (because it only involves the stretching of the time variable due to the departure of the shock velocity from the sonic velocity), is evidently not acceptable for the pressure. It can, however, be immediately eliminated by observing that if one writes

$$(\phi_{1R}+\psi_{1S})\,(S+R) = (\phi_{1R}-\psi_{1S})\,(S-R) + 2R\phi_{1R} + 2S\psi_{1S}$$

the first term is non-secular (because $S-R = 2X$ is limited) and the other two terms are secular but can respectively be absorbed in the arbitrary functions $\phi_2^*(R,\theta)$ and $\psi_2^*(R,\theta)$. Indeed if we write

$$\phi_2 = \phi_2^* - \{[\tfrac{1}{2}(3-\gamma)]\,\langle\phi_1\rangle - \tfrac{1}{2}(\gamma+1)\,\phi_{1m}\}\,R\phi_{1R},$$

$$\psi_2 = \psi_2^* - \{[\tfrac{1}{2}(3-\gamma)]\,\langle\phi_1\rangle - \tfrac{1}{2}(\gamma+1)\,\phi_{1m}\}\,S\phi_{1S},$$

the resulting expressions

$$\left.\begin{aligned} U_2 &= \tfrac{1}{4}[(\gamma+1)(\phi_1-\phi_{1m})\phi_{1R}-2\phi_{1\theta}+(\gamma+1)(\psi_1-\phi_{1m})\psi_{1S}-2\psi_{1\theta}](S-R)\\ &\qquad +[\tfrac{1}{2}(\gamma+1)](\psi_1^2-\phi_1^2)+\phi_2-\psi_2,\\ (P_2/\gamma) &= \tfrac{1}{4}[(\gamma+1)(\phi_1-\phi_{1m})\phi_{1R}-2\phi_{1\theta}-(\gamma+1)(\psi_1-\phi_{1m})\psi_{1S}+2\psi_{1\theta}](S-R)\\ &\qquad -[\tfrac{1}{2}(\gamma+1)](\psi_1-\phi_1)^2+\phi_2+\psi_2,\end{aligned}\right\} \tag{4.26}$$

present no secular variation provided ϕ_2 and ψ_2 are periodic in the respective R or S variable, with the same period 2 as ϕ_1 and ψ_1. The second-order condition obtained from (4.3), $U_2 = 0$ at $X = 0$, is evidently satisfied provided $\psi_2(T,\theta) = \phi_2(T,\theta)$. Application of the condition $U = 0$ at $X = 1$ results in the equation

$$(\gamma+1)(\phi_1-\phi_{1m})\phi_{1R}-2\phi_{1\theta} = 0 \tag{4.27}$$

since at $R = T-1$, $S = T+1$ we have $\phi_1 = \psi_1$ and $\phi_2 = \psi_2$ because of the periodicity.

Equation (4.27) is substantially the inviscid Burgers equation. It determines the shape of the waves following an arbitrary initial disturbance, and their distortion with θ. Its general solution can be written as

$$\phi_1 = H\left\{R+[\tfrac{1}{2}(\gamma+1)]\left(\phi_1\theta-\int_0^\theta \phi_{1m}\,d\theta'\right)\right\}, \tag{4.28}$$

where the arbitrary function H is determined by application of the initial condition

$$\phi_1(R,0) = \phi_{1i}(R)$$

so that $H = \phi_{1i}$.

It is not our purpose here to discuss in detail the consequences of (4.28), but only to present a few observations. First of all, by integrating (4.27) over one period, and observing that $(\phi_1-\phi_{1m})^2$ is continuous by definition even at the shock, we obtain

$$\langle\phi_1\rangle_\theta = 0,$$

with the mean value $\langle\phi_1\rangle$ defined by (4.22). Hence the mean value $2\gamma\langle\phi_1\rangle$ of the pressure perturbation remains constantly equal to its initial value. In particular it vanishes if the initial disturbance does not involve a variation of the mass contained in the pipe. In this case, which we shall adopt for simplicity, we have from (4.25)

$$\omega_1 = \tfrac{1}{2}(\gamma+1)\pi\phi_{1m}. \tag{4.29}$$

The first-order frequency perturbation is, as we see, determined by the mean value of the pressure at the shock (the additional term in (4.25) can be attributed to a variation of the mean sonic velocity due to the initial isentropic variation of the mass content). If the shock is absent of course ϕ_{1m} loses its meaning, or rather can be chosen to be any of the ϕ_1 values. The meaning is simple: the frequency of repetition of any value of ϕ_1 depends on the ϕ_1 level because of the gradual distortion of the wave shape. Hence there is an element of arbitrariness in the determination of the frequency, and we can if we like choose $\omega_1 = 0$. In the shockless case the appropriate equation and the general solution are obtained from (4.27) and (4.28) by substituting for ϕ_{1m} its value determined from (4.34) in terms of ω_1, and by choosing conveniently (but arbitrarily) the value of ω_1, for instance zero.

One can easily derive from (4.28) the known behaviour of the waves following a continuous initial disturbance, and the steepening of their front until the maximum slope ϕ_{1R} becomes infinite after a time $\theta_{sh} = [2/(\gamma+1)](1/\phi_{1iR\,\max})$ (where $\phi_{1iR\,\max}$ is the maximum (positive) slope of the initial $\phi_{1i}(R)$). For $\theta > \theta_{sh}$, ϕ_2 is discontinuous and the wave starts decaying. This can be seen immediately by calculating the behaviour of the total acoustic energy of the wave

$$E_1 = \frac{1}{2}\int_0^1 \{[P_1^2(T,X)/\gamma^2] + U_1^2(T,X)\}\,\mathrm{d}X = 2\langle\phi_1^2\rangle.$$

Multiplying (4.27) by ϕ_1 and integrating over a cycle from shock to shock we have

$$2[\mathrm{d}\langle\phi_1^2\rangle/\mathrm{d}\theta] = (\gamma+1)\int_{R_{sh}}^{R_{sh}+2} [(\phi_1-\phi_{1m})^2+\phi_{1m}(\phi_1-\phi_{1m})]\,\phi_{1R}\,\mathrm{d}R$$
$$= -\tfrac{1}{3}(\gamma+1)\,\Delta(\phi_1-\phi_{1m})^3 - \tfrac{1}{2}(\gamma+1)\,\phi_{1m}\Delta(\phi_1-\phi_{1m})^2,$$

where again $\Delta f = f(R_{sh}) - f(R_{sh}+2)$ is the jump across the shock of any quantity f. The second term vanishes by definition, and the first is immediately calculated in terms of $\Delta\phi_1$ and is equal to $-[(\gamma+1)/12]\,(\Delta\phi_1)^3 = -\Delta\sigma_3$, where $\Delta\sigma_3$ is the jump of the entropy across the shock. Hence we obtain

$$\mathrm{d}E_1/\mathrm{d}\theta = -\Delta\sigma_3,$$

so that E_1 stays constant despite the wave distortion until the shock appears, after which it starts decaying to vanish at $\theta = \infty$. If ϕ_{1i} contains more than one identical wave in a period, modes higher than the fundamental can be produced. Without further labour, these can be derived from the above treatment of the fundamental mode by just assuming that the reference normalization length is the corresponding wavelength, rather than the length of the chamber.

No particular difficulty seems to prevent the extension of the above treatment to higher order, making use of course of the original conservation equations across the shock, provided a new length scale X/ϵ is introduced to account for the fact that the third-order entropy perturbations produced by the shocks must move with the velocity u of the gas, which is of order ϵ. For approximations above the third order both R/ϵ and S/ϵ scales (and hence a third time scale T/ϵ) must be added to take care of the waves originating at the entropy perturbations. Hence we again see that the scale coordinate perturbation technique must be supplemented by the multiple scale technique to take care of higher order approximations in the presence of shocks. We also see that the introduction of the second length and the third time scales needed for higher order approximations is required for other reasons than those leading to the addition of the second time scale. Indeed the second time scale is needed, as in progressive waves, to take care of the quasi-periodicity, and becomes unnecessary when looking for truly periodic solutions (which would be possible, for instance, if the necessary energy input could be supplied by an energy source, for instance chemical reactions); the second length scale and the third time scale originate from the presence in the equations of convective terms, and are required to take care of the entropy convection and of the relative wave reflexions. The need for a second length scale when entropy perturbations are present has already been briefly stated by Lick (1969) for the case when the entropy perturbations are produced by chemical reactions.

5. Resonant oscillations in closed tubes; the inviscid Chester problem

The concepts and the equations of §4 can be applied to study the case of the oscillations generated in a tube by an oscillating piston located at the $X = 1$ end, the $X = 0$ end being still closed. Equations (4.26) still hold with ϕ_{1m} replaced by its expression in terms of $\langle\phi_1\rangle$ and ω_1 as obtained from (4.25) appropriately modified as we shall see. Moreover, the value of ω_1 is related to the normalized piston frequency ω by

$$\omega = N\pi + \epsilon\omega_1 + \dots, \tag{5.1}$$

where N is an integer and $N\pi$ are the frequencies of the acoustic organ pipe oscillations for the various modes ($N = 1$ for the fundamental). Hence ω_1 is prescribed and represents the deviation from the acoustic resonant conditions.

For $N \neq 1$ the modification to be introduced in (4.25) consist simply in replacing ω_1 by ω_1/N. The functions $\phi_1, \psi_1, \phi_2, \psi_2$ must again satisfy the condition

$$\phi_1(T, \theta) = \psi_1(T, \theta); \quad \phi_2(T, \theta) = \psi_2(T, \theta), \tag{5.2}$$

because of the b.c. at $X = 0$. Moreover, they are expected to be periodic in T with the same period $2/N$ as the piston; and, of course, it is again true that

$$\phi_1(T+2, \theta) = \phi_1(T, \theta), \quad \phi_2(T+2, \theta) = \phi_2(T, \theta). \tag{5.3}$$

Hence, again, we have $U_1 = 0$ at $X = 1$. This means that only U_2 can be $\neq 0$ at the piston, that is, the velocity of the piston must be of order ϵ^2. If the piston is assumed to be at rest for $t < 0$ and to move as $l \sin \omega t$ for $t \geqslant 0$, the piston velocity is zero for $t < 0$ and $l\omega \cos \omega t$ for $t \geqslant 0$. We can take therefore $l = \epsilon^2$, so that the piston velocity is, by virtue of (5.1)

$$\epsilon^2 N\pi \cos N\pi T + O(\epsilon^3) + \dots.$$

Hence applying the first (4.26) at $X = 1$, where $S - R = 2$, and using the relations (5.2) and (5.3) we obtain the equation

$$(\gamma+1)\,(\phi_1 - \phi_{1m})\,\phi_{1R} - 2\phi_{1\theta} = N\pi \cos N\pi T,$$

where the arguments of ϕ_1 are $R = T-1$ and θ, and where ϕ_{1m} must satisfy the modified (4.25). It is convenient at this point to switch to the variables

$$\tau = \tfrac{1}{2}N\pi R - \tfrac{1}{4}\pi, \quad \delta = \tfrac{1}{2}N\pi\theta;$$

so that the equation becomes

$$\tfrac{1}{2}(\gamma+1)\,(\phi_1 - \phi_{1m})\,\phi_{1\tau} - \phi_{1\delta} = -\sin 2\tau. \tag{5.4}$$

Integrating upon a period, that is from 2τ to $2\tau + 2\pi$, the first term on the l.h.s. and the r.h.s., provide zero contributions. Hence we again obtain the result that the mean value of ϕ_1 does not change with δ (this constancy expresses again the conservation of mass). If the initial disturbance at $\theta = \delta = 0$ does not involve a change of mass (for instance if the gas is initially at rest) the mean value $\langle\phi_1\rangle$ will remain constantly zero, and the modified (4.25) gives simply

$$\omega_1 = \tfrac{1}{2}N\pi(\gamma+1)\,\phi_{1m}. \tag{5.5}$$

The general solution of (5.4) is

$$h = H[(\gamma+1)^{-\frac{1}{2}} F(\tau, h^{-1}) + h\delta \operatorname{sg}(\phi_1 - \phi_{1m})], \tag{5.6}$$

with

$$h^2 = \tfrac{1}{4}(\gamma+1)\,(\phi_1 - \phi_{1m})^2 + \sin^2\tau. \tag{5.7}$$

Here

$$F(\tau, h^{-1}) = \int_0^\tau (1 - h^{-2}\sin^2\tau')^{-\frac{1}{2}}\,\mathrm{d}\tau'$$

represents the incomplete elliptic integral of the first kind and sg is the sign function:

$$\operatorname{sg} f = \pm 1 \quad \text{for} \quad f \gtrless 0.$$

H is an arbitrary function of the argument in brackets, and is determined by the initial conditions at $\delta = 0$. If, for instance, $\phi_1 = 0$ initially, H is defined by

$$[(\gamma+1)^{-1}(\omega_1/N\pi)^2 + \sin^2\tau]^{\frac{1}{2}} = H[(\gamma+1)^{-\frac{1}{2}} F\{\tau, [(\gamma+1)^{-1}(\omega_1/N\pi)^2 + \sin^2\tau]^{-\frac{1}{2}}\}].$$

It is immediately realized that the quantity h can become < 1, so that the elliptic integral $F(\tau, k)$ needs to be known for all values of k (in the range of τ where it is real, of course) and not only for $k < 1$, as usual.

The discussion of (5.6), (5.7) is substantially more complicated than that of (4.28), particularly because of the necessity of choosing the sign of the $\phi_1 - \phi_{1m}$ associated with any value of h. Without presenting the discussion, it is sufficient to say that, no matter what the initial distribution of ϕ_1 is, h tends to become constant for $\delta \to \infty$. The asymptotic value of h and the asymptotic behaviour of ϕ_1 depends on the deviation ω_1 from the resonant frequency. They can be directly obtained by setting $\phi_{1\delta} = 0$ in (5.4), and integrating. It is found that for

$$|\omega_1| \geqslant 2\sqrt{(\gamma+1)},$$

ϕ_1 is continuous and h is the root (always $\geqslant 1$) of the equation

$$hE(\tfrac{1}{2}\pi, h^{-1}) = |\omega_1|/2\sqrt{(\gamma+1)}, \tag{5.8}$$

where $E(\frac{1}{2}\pi, k) = \int_0^{\frac{1}{2}\pi} (1-k^2\sin^2\tau)^{\frac{1}{2}}\, d\tau$ is the complete elliptic integral of the second kind. Indeed when this equation is satisfied, the value $\langle\phi_1\rangle$ obtained from (5.7) vanishes as it should.

When $|\omega_1| < 2\sqrt{(\gamma+1)}$ (near resonant conditions) the value of h is always 1 and from (5.7) one obtains for $\alpha \leqslant \tau \leqslant \alpha+\pi$

$$\tfrac{1}{2}\sqrt{\gamma+1}\,\phi_1 = (2/\pi)\sin\alpha + \cos\tau,$$

with

$$\sin\alpha = \omega_1/2\sqrt{(\gamma+1)}, \quad \cos\alpha > 0,$$

the last inequality being required by the need of eliminating negative shocks. Indeed the values of ϕ_1 are repeated periodically with period π in τ, so that for all τ's one can write

$$\tfrac{1}{2}\sqrt{\gamma+1}\,\phi_1 = (2/\pi)\sin\alpha + \cos\tau\,\mathrm{sg}\,[\sin(\tau-\alpha)], \tag{5.9}$$

and at α, $\alpha\pm\pi, \ldots,$ ϕ_1 presents shock discontinuities. Again the condition $\langle\phi_1\rangle = 0$ is satisfied for any period by virtue of the presence of the sign function.

The asymptotic solutions just discussed coincide with those obtained by Chester (1944). As we have derived then, their validity is confined to $\epsilon\omega_1 \ll N\pi$ (without limit on ω_1 itself), a condition which is certainly satisfied in the near resonant region $|\omega_1| < 2\sqrt{(\gamma+1)}$. Outside this region (that is away from resonance) $|\omega_1|$ can be large, and we see from (5.5) that ϕ_{1m} can be $\gg 1$. Neglecting ϕ_1 in front of ϕ_{1m}, integration of (5.4) with $\phi_{1T} = 0$ provides the asymptotic linear result

$$\phi_1 = -(N\pi/2\omega_1)\cos 2\tau = -(-1)^N(N\pi/2\omega_1)\sin\omega t.$$

Incidentally Chester, after observing that this solution does not agree, except for small $\epsilon\omega_1$, with the exact acoustic solution

$$\phi_1 = -(\epsilon\omega/2\sin\omega)\sin\omega t$$

in an attempt to make the nonlinear solution uniformly valid for all ω's, has suggested replacing everywhere in the treatment above $(-1)^N$ by $\cos\omega$ and $\epsilon\omega_1$ by $(N\pi/\omega)\tan\omega$, a substitution that indeed does not alter the results for $\epsilon\omega_1 \ll 1$ and reconciles the last two expressions for ϕ_1. We observe here that the same result can be obtained by replacing $(-1)^N$ by $\mathrm{sg}\,(\cos\omega)$ and $\epsilon\omega_1$ by $(N\pi/\omega)\,[\mathrm{sg}\,(\cos\omega)]\sin\omega$. The two solutions agree for small $\epsilon\omega_1$, and hence in the shock region. In the shockless region, however, where $\omega - N\pi$ can be large, the values of h as obtained from (5.8) are different, and hence the two solutions for ϕ_1 are also different. Consequently, neither can be considered a uniformly valid approximation for all ω's.

At resonance the solution (5.9) becomes (with $\omega_1 = \alpha = 0$)

$$\tfrac{1}{2}(\gamma+1)^{\frac{1}{2}}\phi_1 = \cos\tau\,\mathrm{sg}\,(\sin\tau). \tag{5.10}$$

Chester observed that, if we relieve the condition that the period of ϕ_1 and of the piston must coincide, the periodicity condition (5.3) may still be satisfied for even N by

$$\tfrac{1}{2}(\gamma+1)^{\frac{1}{2}}\phi_1 = \cos\tau, \tag{5.11}$$

which is indeed a solution of (5.4) when $\phi_{1m} = 0$. Chester was able to rule out this solution only by extrapolation from the corresponding solution of the viscous problem. Here we observe that the introduction of the second time scale allows the examination of the stability of the solutions, showing that, contrary to (5.10), (5.11) is an unstable solution.

To show this we study the behaviour of a small disturbance η taking

$$\tfrac{1}{2}(\gamma+1)^{\frac{1}{2}}\phi_1 = \cos\tau + \eta(\tau, \delta).$$

Substituting in (5.4) and linearizing we get

$$(\cos\tau)\,\eta_\tau - (\gamma+1)^{-\frac{1}{2}}\eta_\delta - (\sin\tau)\,\eta = 0.$$

The general solution of this equation is

$$|\cos\tau|\eta(\tau, \delta) = H\{[(1+\sin\tau)/(1-\sin\tau)]\exp[2(\gamma+1)^{-\frac{1}{2}}\delta]\}, \tag{5.12}$$

where H is an arbitrary function of the argument in braces, to be determined by the condition that at $\delta = 0$, $\eta(\tau, 0) = \eta_i(\tau)$. Hence we obtain

$$|\cos\tau|\,\eta_i(\tau) = H[(1+\sin\tau)/(1-\sin\tau)].$$

But if we define
$$\sin z = \frac{1+\sin\tau-(1-\sin\tau)\exp[-2(\gamma+1)^{-\frac{1}{2}}\delta]}{1+\sin\tau+(1-\sin\tau)\exp[-2(\gamma+1)^{-\frac{1}{2}}\delta]}$$

we obtain from the last two equations

$$|\cos\tau|\,\eta(\tau, \delta) = H[(1+\sin z)/(1-\sin z)] = |\cos z|\,\eta_i(z),$$

and hence
$$\eta(\tau, \delta)/\eta_i(z) = |\cos z|/|\cos\tau|$$
$$= \frac{2\exp[-(\gamma+1)^{-\frac{1}{2}}\delta]}{1+\sin\tau+(1-\sin\tau)\exp[-2(\gamma+1)^{-\frac{1}{2}}\delta]}. \tag{5.13}$$

This equation shows that in addition to a shift from τ to z, any value of the initial disturbance is modified by a factor given by the second member. Clearly this factor does, for a given τ, reach a maximum value $|\cos\tau|^{-1}$ at $\sin z = 0$, that is for a value of δ which is negative if $\sin\tau > 0$ (where the factor steadily decays for $\delta > 0$), but positive when $\sin\tau < 0$ and increasingly large as $\sin\tau \to -1$. We see that with increasing δ the disturbance tends to become more and more concentrated around $\sin\tau = -1$, until at $\delta = \infty$ it is entirely concentrated at $\sin\tau = -1$ with an infinite amplitude. We conclude that the solution (5.11) is unstable. If, on the contrary, we consider a disturbance η from the solution (5.10) and we follow the same treatment, we find for η the same expressions (5.12) or (5.13) with $\sin\tau$ replaced by $|\sin\tau|$ in the second members. In this case the second member of (5.13) always decays with increasing δ and the solution (5.10) is stable.

From the above equations the pressure on the piston face can be derived, and the work supplied over a cycle by the piston to the gases calculated. Since the piston velocity is of $O(\epsilon^2)$ and the pressure perturbation of $O(\epsilon)$, the work is of $O(\epsilon^3)$, and so is the corresponding power necessary to drive the piston. It is easily found that this power is zero in the shockless region away from resonance, but is finite and proportional to $\cos^3\alpha$ in the near resonant region, being maximum at resonance, $\alpha = 0$. This power is, of course, dissipated by the shocks, and for adiabatic wall will entrain a secular variation of entropy, pressure, temperature, sonic velocity and resonant conditions starting with the second-order approximation since indeed the work dissipated is proportional to $\epsilon^3 T = \epsilon^2\theta$.

In addition to presenting the secular variation beginning with the second-order approximation, and therefore requiring the introduction of the second time scale even in the case when the asymptotic first-order approximation becomes independent of θ, the determination of higher order approximations would again require, as for the example of § 4, the introduction of additional scales for R and S to take care of the convection of the entropy perturbation due to the shock and of the consequent wave reflexions.

6. Discussion

The following observations and conclusions can be extracted from the above examples.

(1) The procedure which allows by-passing of the transformed equations substantially simplifies the application of the coordinate perturbation technique, thus allowing easy extension to more complicated problems.

(2) It appears that no advantage whatsoever, and only the disadvantage of complication and confusion can be expected from the use of the transformed differential equations.

(3) Contrary to Lighthill's original suggestion and to Pritulo's formulation, the perturbation should be applied to all independent variables.

(4) In addition to their being uniformly valid, the expansions in the perturbed co-ordinates are formally much simpler than in the unperturbed coordinates, a result which may be important for higher approximations and tends to show that the perturbed variables are the natural co-ordinates to use.

(5) In gasdynamical problems containing shocks the coordinate perturbation technique must be supplemented (in case higher approximations are desired) by the multiple scale technique. This makes it possible indeed to take care of the flow features originating from the lack of parallelism of the nonlinear characteristics or from the presence of shocks and of entropy and vorticity perturbations.

(6) There are cases when three (or more) scales may be needed for the same variable. Hence it seems that the correct name should be the 'multiple scale' or 'multiple variable' method, rather than 'double scale' or 'two variables', most often used today. Of course, reference is not made here to existing procedures where an infinite number of scales are used for the same variable, in lieu of straining. The scales discussed in this paper correspond to a distinct physical phenomenon and cannot be taken care of by way of straining.

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